Journal of Statistical Physics, Vol. 66, Nos. 3/4, 1992

A Note on Differentiability of the Cluster Density for Independent Percolation in High Dimensions

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Received February 12, 1991; final September 26, 1991

The cluster density function of independent percolation κ in a d-dimensional lattice is considered. For each n, it is shown that $\kappa(p)$ has finite nth left-derivative at critical probability p_c if d is sufficiently large. This result agrees with the Bethe lattice approximation, where the nth one-sided derivative of $\kappa(p)$ is bounded at p_c for all n.

KEY WORDS: Percolation; critical exponent; number of clusters per vertex.

1. INTRODUCTION

We consider percolation in a d-dimensional lattice Z^d. Let B be the set of all bonds where each bond connects two nearest-neighbor lattice sites in Z^d. Suppose these bonds are open or closed independently, and for each $b \in B$, the probability that b is open is equal to p and the probability that b is closed is equal to $1-p$. Here p is a constant. The underlying probability measure for this model will be denoted by P_p and the expectation with respect to P_p denoted by E_p. Given a configuration of open or closed bonds, two lattice sites x, y in Z^d are said to be connected if there exists a path consisting of open bonds which connects x and y. If $x = y$, then x, y are also said to be connected. The connectedness gives an equivalence relation on the set of all lattice sites on Z^d, and Z^d can be decomposed into connected components. Each component is called a cluster. A cluster is called an infinite cluster if it has infinitely many lattice sites. Let $\phi(p)$ be the probability that there exists an infinite cluster. It is well known (see, e.g., refs. 1 and 2) that for $d \geqslant 2$, there exists $0 < p_c < 1$ such that $\phi(p) = 0$ for all $p < p_c$ and $\phi(p) = 1$ for all $p > p_c$.

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There are some other quantities related to this model which possess discontinuities when p varies. In this paper, we consider $\kappa(p) = E_p(|C_0|^{-1})$, where C_0 is the cluster containing the origin of Z^d and $|C_0|$ is the cardinality of C_0. The function $\kappa(p)$ is also known as the number of open clusters per vertex (see, e.g., refs. 1 and 2). Here, we shall call $\kappa(p)$ the cluster density of percolation. The cluster density is known to be continuously differentiable on $[0, 1]$, analytic on $[0, p_c)$, and infinitely differentiable on $(p_c, 1]$. Kesten[5] proved that, for $d=2$, $\kappa(p)$ is twice differentiable at p_c. There are no other rigorous results concerning the differentiability of $\kappa(p)$. In the case of percolation on Bethe lattices (binary tree), however, the cluster density $\kappa_B(p)$ can be computed explicitly,

$$\kappa_B(p) = \frac{1+2\varepsilon}{1-2\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \binom{2n}{n-1} \frac{1}{4^n} (1-4\varepsilon^2)^n \tag{1.1}$$

with $\varepsilon = -p + \frac{1}{2}$. It is easy to see from (1.1) that $\kappa_B(p)$ is twice differentiable on $[0, 1]$, $\kappa_B(p)$ is analytic in $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, and $\kappa_B^{(3)}(p)$ has a jump discontinuity at $p_c = \frac{1}{2}$. Moreover, for $n \geq 3$, $\kappa_B^{(n)}(p)$ is bounded, as $p \to p_c$. Boundedness of $\kappa_B^{(n)}(p)$ as $p \to p_c$ also holds for a Bethe lattice with arbitrary degrees. For the convenience of the reader, we include a proof of this fact and an explicit formula for κ_B in the Appendix.

In the physics literature, percolation on Bethe lattices is considered as an infinite-dimensional approximation of percolations on Z^d. Thus, it is natural to think that the same situation happens in percolations on Z^d if d is above a certain critical dimension d_c. In other words, $\kappa^{(n)}(p)$ is bounded as $p \to p_c$. It is actually conjectured in refs. 2 and 3 that

$$\kappa^{(n)}(p) \approx |p - p_c|^{-1-\alpha_3} \qquad \text{as} \quad p \to p_c \tag{1.2}$$

$-1 < \alpha_3 < 0$ for $2 \leq d \leq 5$ and $\alpha_3 = -1$ for all $d \geq 6$. In this paper, we obtain the following result.

Theorem 1.1. Let $\kappa(p)$ be the cluster density for percolation of Z^d. For any n, there exists d_n such that $\kappa(p)$ is continuously n times differentiable on $[0, p_c]$ if $d \geq d_n$. Here the derivatives at p_c mean left-derivatives. In particular, for any n, $|\kappa^{(n)}(p)|$ is uniformly bounded on $[0, p_c]$ if d is sufficiently large.

Theorem 1.1 implies that if $-1 \leq \alpha_3 < 0$, then $\alpha_3 = -1$ at sufficiently high dimensions. This result agrees with the conjecture made in refs. 2 and 3.

Our estimate of d_n is $d_n = \max\{d_0, 4n-3\}$, where d_0 is the critical dimension such that the infrared bound (3.13) holds for $d \geq d_0$. It follows from ref. 4 that (3.13) holds for $d \geqslant d_0$ with $d_0 = 48$. It is believed in the physics literatures that $d_0 = 7$. Our method works also for a spread-out model (see, e.g., ref. 4) and Theorem 1.1 holds for a spread model. In this case, d_0 can be chosen to be 7. It is generally believed in the physics literature that the spread-out model and the nearest-neighbor model belong to the same universality class and they have the same critical dimension.

Our estimate $d_n = \max\{d_0, 4n-3\}$ is not sharp. The only criterion is that $d_n \geqslant d_0$ and that the Feynman diagrams appearing in $K^{(n)}$ converge. For example, when $n = 3$, d_3 can be chosen as $\max(d_0, 7)$. We have not found the best possible way to estimate all the Feynman diagrams appearing in our calculations.

Our proof of Theorem 1.1 depends on a new formula of Russo's type (Theorem 2.1). We also develop a systematic way to handle various types of pivotal bounds (Proposition 2.4) which appear in this formula.

2. RUSSO'S FORMULA AND ITS VARIATIONS

We shall use Russo's formula[7] to prove Theorem 1.1. Here we present Russo's formula in a slightly different way (Theorem 2.1) so that (1) it includes also the case that the event is not necessarily increasing or decreasing, and (2) it allows one to take higher derivatives. Let Λ be a finite set and $\{w(b),\ b \in \Lambda\}$ a family of independent Bernoulli random variables with $P(w(b) = 1) = p_b$ and $P(w(b) = 0) = 1 - p_b$. Given any $a \in \Lambda$ and a configuration $w = \{w(b);\ b \in \Lambda\}$, configurations w_a^+, w_a^- are defined to be $w_a^+(b) = w_a^-(b) = w(b)$ for all $b \neq a$, $w_a^+(a) = 1$ and $w_a^-(a) = 0$. Given $b \in \Lambda$ and an event A, the event that b is positively pivotal for A is defined by $A_b = \{w;\ w_b^+ \in A \text{ and } w_b^- \notin A\}$. Similarly, the event that b is negatively pivotal for A is defined by $A^b = \{w;\ w_b^+ \notin A \text{ and } w_b^- \in A\}$. Note that $A_b \cap A^b = \varnothing$. The event that b is pivotal for A is defined as $A(b) = A_b \cup A^b$. It follows from the above definitions that

$$
\begin{aligned}
I_{A(b)}(w) &= I_A(w_b^+)\, I_{A^c}(w_b^-) + I_A(w_b^-)\, I_{A^c}(w_b^+) \\
I_{A_b}(w) &= I_A(w_b^+)\, I_{A^c}(w_b^-) \qquad\qquad (2.1) \\
I_{A^b}(w) &= I_A(w_b^-)\, I_{A^c}(w_b^+)
\end{aligned}
$$

If $w \in A_b$, then $w_b^+, w_b^- \in A_b$. This also holds for A^b and $A(b)$. Therefore

$$A_b,\ A^b,\ A(b) \quad \text{are measurable with respect to } \sigma\{w(a);\ a \in \Lambda,\ a \neq b\} \qquad (2.2)$$

A partial ordering on the set of configurations is defined by $w \leqslant w'$ if $w(b) \leqslant w'(b)$ for all $b \in \Lambda$. An event A is said to be increasing if $I_A(w) \leqslant$

$I_A(w')$ whenever $w \leqslant w'$. An event A is decreasing if A^c is increasing. From these definitions it follows that

$$I_{A(b)}(w) = I_A(w_b^+)\, I_{A^c}(w_b^-) = I_{A_b}(w) \qquad \text{if } A \text{ is increasing} \tag{2.3}$$

Finally, we note that if A is increasing, then

$$A(b) = \alpha \cap \beta \tag{2.4}$$

where α is increasing and β is decreasing. Indeed, let α, β be events such that $I_A(w_b^+) = I_\alpha(w)$ and $I_{A^c}(w_b^-) = I_\beta(w)$. Then α is increasing, β is decreasing, and (2.4) follows from (2.3). Using the above notations, we can put Russo's formula in the following form.

Theorem 2.1. For any event A, and $b \in \Lambda$,

$$\frac{dP(A)}{dp_b} = P(A_b) - P(A^b) \tag{2.5}$$

Corollary 2.2. If A is increasing, then $dP(A)/dp_b = P(A(b))$. If A is decreasing, then $dP(A)/dp_b = -P(A(b))$. For any event A, $|dP(A)/dp_b| \leqslant P(A(b))$.

Remark 2.3. Theorem 2.1 allows one to take higher derivatives of $P(A)$. For example,

$$\frac{d}{dp_{b_2}}\left(\frac{dP(A)}{dp_{b_1}}\right) = P(A_{b_1,b_2}) - P((A_{b_1})^{b_2}) - P((A^{b_1})_{b_2}) + P(A^{b_1,b_2}) \tag{2.6}$$

for any event A, $b_1, b_2 \in \Lambda$. In particular, if A is increasing, then

$$\frac{d}{dp_{b_2}}\left(\frac{dP(A)}{dp_{b_1}}\right) = P(A_{b_1,b_2}) - P((A_{b_1})^{b_2}) \tag{2.7}$$

Corollary 2.2 is a immediately consequence of Theorem 2.1 and (2.3). To prove Theorem 2.1, we write

$$P(A) = \sum_w I_A(w) \prod_{a \in \Lambda} p_a^{w(a)}(1-p_a)^{1-w(a)} \tag{2.8}$$

Then

$$\frac{dP(A)}{dp_b} = \sum_w I_A(w)[I_{w(b)=1}(w) - I_{w(b)=0}(w)] \times \sum_{a \in \Lambda, a \neq b} p_a^{w(a)}(1-p_a)^{1-w(a)} \tag{2.9}$$

Let ζ denote a configuration on $A\backslash b$ and $\zeta \cup \eta$ a configuration on A such that $(\zeta \cup \eta)(a) = \zeta(a)$ for all $a \in A\backslash b$ and $(\zeta \cup \eta)(b) = \eta$. Then (2.9) can be written as

$$\frac{dP(A)}{dp_b} = \sum_{\zeta} \left[I_A(\zeta \cup 1) - I_A(\zeta \cup 0) \right] \times \prod_{a \in A, a \neq b} p_a^{\zeta(a)} (1 - p_a)^{1 - \zeta(a)} \tag{2.10}$$

By (2.2), I_{A_b} and I_{A^b} are functions of ζ and

$$\begin{aligned} I_A(\zeta \cup 1) - I_A(\zeta \cup 0) &= I_A(\zeta \cup 1)\, I_{A^c}(\zeta \cup 0) - I_{A^c}(\zeta \cup 1)\, I_A(\zeta \cup 0) \\ &= I_{A_b}(\zeta) - I_{A^b}(\zeta) \end{aligned}$$

This and (2.10) imply Theorem 2.1.

We shall also use the following variation of Russo's formula, which first appeared in ref. 6: If A is increasing and B is decreasing, then

$$\frac{dP(A \cap B)}{dp_b} = P(A(b) \cap (B(b))^c \cap B) - P(B(b) \cap (A(b))^c \cap A)$$

(2.11) can be easily proved by using Theorem 2.1 and (2.1).

Formulas (2.1) and (2.11) can be applied to percolation in Z^d by associating with each bond a random variable $w(b)$, where $w(b) = 1$ if b is open and $w(b) = 0$ if b is closed. However, after several derivatives, the relations between pivotal bonds of connectedness of certain lattice sites become complicated. In order to handle them in an efficient way, we extend the definition of pivotal bonds as follows. First we consider a graph consisting of nodes and edges where each edge joins two different nodes. Nodes are represented by circles and they are labeled by subsets of $Z^d, \Gamma_1, ..., \Gamma_l$. Edges are labeled by bonds in $Z^d, b_1, b_2, ..., b_s$. [Here we are using nodes and edges as in the terminology of general graph theory (see, e.g., ref. 1). Suppose in our graph an edge b joins nodes Γ_α and Γ_β; it does not imply that b connects set Γ_α and Γ_β in the Z^d lattice.]

Given a configuration w of open and closed bonds in Z^d and $x \in Z^d$, let $C(x) = C(x, w)$ be the cluster that contains x. For a subset Γ of Z^d, we let $C(\Gamma) = C(\Gamma, w) = \bigcup_{x \in \Gamma} C(x, w)$. Two subsets Γ_1 and Γ_2 of Z^d are said to be connected if there are $x \in \Gamma_1$ and $y \in \Gamma_2$ such that x and y are connected by a path of open bonds of w. A set Γ is said to be connected if Γ is contained in a cluster. Given a graph G with nodes $\Gamma_1, \Gamma_2, ..., \Gamma_l$ and edges $b_1, b_2, ..., b_s$, the event that $b_1, b_2, ..., b_s$ are pivotal bonds for the connection of $\Gamma_1, ..., \Gamma_l$ with respect to G, denoted by $B(G)$, is defined by the set of all configurations w such that:

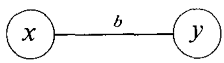
(i) $C(\Gamma_1, w^-)$, $C(\Gamma_2, w^-)$,..., $C(\Gamma_l, w^-)$ are disjoint, where w^- is the configuration obtained from w by setting $w(b_i) = 0$ for all i.

(ii) Upon setting $w(b_i) = 1$ for all i, every b_i can be used to connect $C(\Gamma_\alpha, w^-)$ and $C(\Gamma_\beta, w^-)$, where Γ_α and Γ_β are the nodes that meet the edge b_i in the graph G.

Another useful event is defined by

$$I(G) = \{w;\ C(\Gamma_i, w^-) \text{ is contained in one cluster of } w^-,\ \text{for every } i = 1, 2,..., l\}$$

Note that if G is given by



then $B(G)$ is the event that b is pivotal for the connection of x and y as defined before.

The following special case will be useful in this paper. Let G be a graph consisting of nodes labeled by $\Gamma_1, \Gamma_2,..., \Gamma_l$ and edges $b_1, b_2,..., b_s$. Here $b_i = (x_i, y_i)$, $i = 1, 2,..., s$, and all the b_i are different. Suppose $\Gamma_1, \Gamma_2,..., \Gamma_l$ is a partition of $\{x_i, y_i, i = 1, 2,...\}$ such that $\Gamma_i \neq \varnothing$ and x_i, y_i do not belong to the same Γ_α, for any α. Suppose also that b_i joins Γ_α and Γ_β if $x_i \in \Gamma_\alpha$ and $y_i \in \Gamma_\beta$ (see Fig. 1).

Proposition 2.4. Let G be a graph satisfing the above conditions. Put $I = I(G)$ and $B = B(G)$. Then

$$\frac{d}{dp} P(I \cap B) = \sum_{b \neq b_i \text{ for any } i} \sum_{G'} P(I(G') \cap B(G')) - \sum_{b \neq b_i \text{ for any } i} \sum_{G''} P(I(G'') \cap B(G'')) \tag{2.12}$$

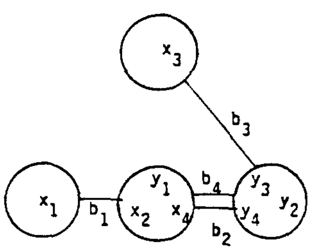


Fig. 1

Here G' runs over all possible graphs of the following type. G' has nodes labeled $\Gamma_1, \Gamma_2,..., \Gamma_{i-1}, \Gamma_i', \Gamma_i'', \Gamma_{i+1},..., \Gamma_l$ and edges $b_1,..., b_s$, $b=(x, y)$ such that $x \in \Gamma_i'$, $y \in \Gamma_i''$, $\Gamma_i' \backslash \{x\} \neq \emptyset$, $\Gamma_i'' \backslash \{y\} \neq \emptyset$, and $\Gamma_i' \cup \Gamma_i'' \backslash \{x, y) = \Gamma_i$. The summation G'' is taken over all G'' of the following type. G'' has nodes labeled by $\Gamma_1, \Gamma_2,..., \bar{\Gamma}_\alpha, \Gamma_{\alpha+1},..., \bar{\Gamma}_\beta, \Gamma_{\beta+1},..., \Gamma_l$ and bonds $b=(x, y)$, $b_1, b_2,..., b_s$ such that

$$\bar{\Gamma}_\alpha = \Gamma_\alpha \cup \{x\}, \qquad \bar{\Gamma}_\beta = \Gamma_\beta \cup \{y\}$$

Proof. The event I is increasing and B is decreasing. Applying (2.12), we get

$$\frac{d}{dp} P(I \cap B) = \sum_b P(I(b) \cap (B(b))^c \cap B) - \sum_b P(B(b) \cap (I(b))^c \cap I) \tag{2.13}$$

Note that $I(b) \cap (B(b))^c \cap B = \bigcup_{G'} \{I(G') \cup B(G')\}$ for $b \neq b_i$ for any i, and $\emptyset$ if $b = b_i$ for some i. For the second sum in (2.13), note that $B(b) \cap (I(b))^c \cap I = \bigcup_{G''} \{I(G'') \cap B(G'')\}$ for $b \neq b_i$ for any i, and $\emptyset$ if $b = b_i$ for some i. Then proposition follows from (2.11) and the observation that both unions over G' and G'' are unions of disjoint events.

3. PROOF OF THEOREM 1.1

Let $B(n) = [-n, n]^d \cap Z^d$, $d \geqslant 2$. Any two lattice sites $x, y \in B(n)$ are said to be connected in $B(n)$ if either $x = y$ or there exists a path γ consisting of open bonds such that γ connects x and y and $\gamma \subseteq [-n, n]^d$. Connectedness in $B(n)$ defines an equivalence relation in $B(n)$ and it decomposes $B(n)$ into connected components. Each connected component is called a cluster in $B(n)$. Let M_n be the number of clusters in $B(n)$. It follows from Grimmett[3] and Wierman[8] that

$$\lim_{n \to \infty} \frac{1}{|B(n)|} M_n = \kappa(p) \text{ a.s. } P_p$$

for all $0 \leqslant p \leqslant 1$. Let $K_n = E(M_n)$. Then, by the Dominated Convergence theorem,

$$\lim_{n \to \infty} \frac{1}{|B(n)|} K_n(p) = \kappa(p) \tag{3.1}$$

By the Ascoli–Arzella theorem and a standard argument using uniform convergence, to prove Theorem 1.1, it is sufficient to prove that for each N, there exists a constant $d_0(N)$ such that for all $d \geqslant d_0(N)$, there exists a constant $C_N(d)$ such that

$$\frac{1}{|B(n)|} |K_n^{(N)}(p)| \leqslant C_N \left(\frac{1}{p}\right)^N \tag{3.2}$$

for all n, all $0 < p \leqslant p_c$.

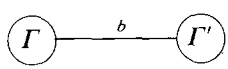
We start to be taking the first derivative of K_n,

$$K_n = EM_n = \sum_{l=1}^{\infty} P(M_n \geqslant l)$$

The event $\{M_n \geqslant l\}$ is decreasing, by Russo's formula,

$$\begin{aligned}
K_n'(p) &= -\sum_{l=1}^{\infty} \sum_{b:\text{ bond in } B(n)} P(\{M_n \geqslant l\}(b)) \\
&= -\sum_{b:\text{ bond in } B(N)} \sum_{l=2}^{\infty} P\{M_n = l-1 \text{ if } w_b = 1 \text{ and} \\
&\qquad\qquad M_n = l \text{ if } w_b = 0\} \\
&= -\sum_{b:\text{ bond in } B(n)} P\{b \text{ is pivotal for the connection} \\
&\qquad\qquad \text{of } x \text{ and } y, \text{ where } x, y \text{ are the end sites of } b\} \\
&= -\frac{1}{2} \sum_{x, x'} P\{B(G_1)\} \qquad (3.3)
\end{aligned}$$

where G_1 is the graph given by

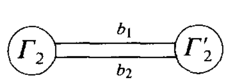


with $\Gamma = \{x\}$ and $\Gamma' = \{x'\}$, and $b = (x, x')$ is a bond.

Note that $B(G_1)$ is a decreasing event. Applying (2.10), we get

$$K_n'' = \frac{1}{2} \sum_{x_2, x_1', x_2, x_2' \in B(n)} P(I_2 \cap B_2) \tag{3.4}$$

where $B_2 = B(G_2)$, $I_2 = I(G_2)$, and G_2 is given by

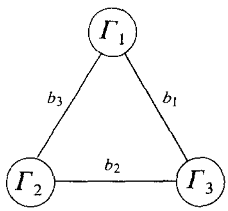


Here $\Gamma_2 = \{x_1, x_2\}$, $\Gamma_2' = \{x_1', x_2'\}$, $b_1 = (x_1, x_1')$, and $b_2 = (x_2, x_2')$ such that $b_1 \neq b_2$.

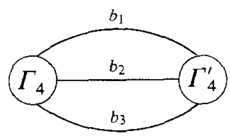
To differentiate K_n'', we note that I_2 is an increasing event and B_2 is a decreasing event. By (2.10),

$$K_n^{(3)} = \sum_{x_1, x_1', x_2, x_2', x_3, x_3' \in B(n)} P(I_{3,1} \cap B_{3,1}) - \frac{1}{2} \sum_{x_1, x_1', x_2, x_2', x_3, x_3' \in B(n)} P(I_{3,2} \cap B_{3,2}) \tag{3.5}$$

where $B_{3,i} = B(G_{3,i})$, $I_{3,i} = I(G_{3,i})$, and $G_{3,i}$ is given by



$G_{3,1}$



$G_{3,2}$

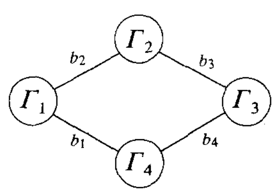
Here $\Gamma_1 = \{x_1, x_3\}$, $\Gamma_2 = \{x_3', x_2\}$, $\Gamma_3 = \{x_1', x_2'\}$, $b_i = (x_i, x_i')$ for $i = 1, 2, 3$, and the b_i are all different; in $G_{3,2}$, $\Gamma_4 = \{x_1, x_2, x_3\}$, $\Gamma_4' = \{x_1', x_2', x_3'\}$.

The events $I_{3,i}$ are again increasing and the $B_{3,i}$ decreasing. We may apply (2.10) again to obtain the fourth derivative of K_n:

$$\begin{aligned} K_n^{(4)} = 3 &\sum_{x_1, x_1', x_2, x_2', x_3, x_3', x_4, x_4'} P(I_{4,1} \cap B_{4,1}) \\ -3 &\sum_{x_1, x_2, x_3, x_4, x_1', x_2', x_3', x_4'} P(I_{4,2} \cap B_{4,2}) \\ -3 &\sum_{x_1, x_2, x_3, x_4, x_1', x_2', x_3', x_4'} P(I_{4,3} \cap B_{4,3}) \\ +\frac{1}{2} &\sum_{x_1, x_2, x_3, x_4, x_1', x_2', x_3', x_4'} P(I_{4,4} \cap B_{4,4}) \end{aligned} \tag{3.6}$$

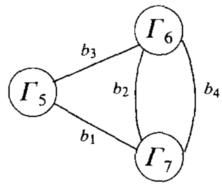
where $B_{4,i} = B(G_{4,i})$, $I_{4,i} = I(G_{4,i})$, and the $G_{4,i}$ are given by

$G_{4,1}$:



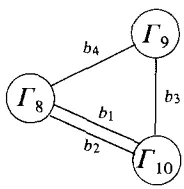
$$\Gamma_5=\{x_1, x_3\},\ \Gamma_6=\{x_3', x_2, x_4\},\ \Gamma_7=\{x_1', x_2', x_4'\};$$

$G_{4,2}$:



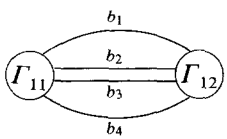
$$\Gamma_8=\{x_1, x_2, x_4\},\ \Gamma_9\{x_4', x_3\},\ \Gamma_{10}=\{x_1', x_2', x_3'\};$$

$G_{4,3}$:



$$\Gamma_1=\{x_1, x_2\},\ \Gamma_2=\{x_2', x_3\},\ \Gamma_3=\{x_3', x_4'\},\ \Gamma_4=\{x_1, x_4\};$$

$G_{4,4}$:



$$\Gamma_{11}=\{x_1, x_2, x_3, x_4\},\ \Gamma_{12}=\{x_1', x_2', x_3', x_4'\},\ b_i=(x_i, x_i').$$

In principle, $K_n^{(N)}$ can be computed by using (2.10) repeatedly. There is a pattern to write down what kind of graphs are in $K_n^{(N)}$. Rather than writing down the explicit formula for $K_n^{(N)}$, we shall start to estimate $K_n^{(N)}$, for $N=1, 2,\ldots$. Using $P(B(G_1))\leqslant 1$ and (3.3), we have

$$\frac{1}{|B(n)|}\,|K_n'(p)|\leqslant d \tag{3.7}$$

for all $0\leqslant p\leqslant 1$.

To estimate K_n'', we note that $I_2\cap B_2\subseteq Q_2$, where Q_2 is the event that there exist two disjoint paths of open bonds such that one path connects x_1 and x_2 and the other path connects x_1' and x_2'. By the v.d. Berg–Kesten inequality,

$$\begin{aligned} P(Q_2)&\leqslant P\{x_1 \text{ is connected to } x_2\}\ P\{x_1' \text{ is connected to } x_2'\}\\ &\leqslant\frac{1}{p^2}\,P\{x_1 \text{ is connected to } x_2\}^2 \end{aligned}\tag{3.8}$$

Hence

$$\frac{1}{|B(n)|}|K_n''| \leqslant \frac{1}{2}\left(\frac{2d}{p}\right)^2 \frac{1}{|B(n)|} \sum_{x_1, x_2 \in B(n)} P\{x_1 \text{ is connected to } x_2\}^2$$

$$\leqslant \frac{1}{2}\left(\frac{2d}{p}\right)^2 \sum_{x \in Z^d} P\{0 \text{ is connected to } x\}^2$$

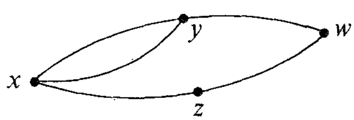
$$= \frac{1}{2}\left(\frac{2d}{p}\right)^2 \sum_{x \in Z^d}$$

$$= \frac{1}{2}\left(\frac{2d}{p}\right)^2 \qquad (3.9)$$

The diagrams are defined as follows. Let $g(x, y)$ denote the probabity that x is connected to y. A Feynman diagram is a graph consisting of vertices and edges. If the vertices of a Feynman diagram F are labeled by $x_1, x_2, \ldots \in Z^d$, then it represents a function

$$\prod_{e \in F} g(x_e, y_e) \qquad (3.10)$$

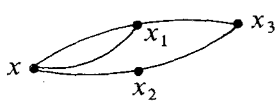
where e is taken over all edges in F, and x_e and y_e are the vertices that meet e. For example,



$$= g^2(x, y)\, g(y, w)\, g(x, z)\, g(z, w)$$

An unlabeled Feynman diagram represents a summation over all labeled Feynman diagrams with labels in Z^d. For example,

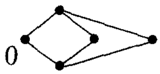
$$= \sum_{x_1, x_2, x_3}$$



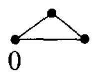
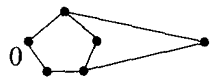
$$= \sum_{x_1, x_2, x_3} g^2(x, x_1)\, g(x, x_2)\, g(x_1, x_3)\, g(x_2, x_3)$$

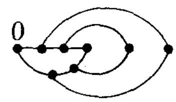
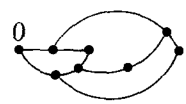
To estimate K_n''', we first note that $I_{3,1} \cap B_{3,1} \subseteq Q_{3,1}$, where $Q_{3,1}$ is the event that there exist three disjoint paths of open bonds such that one connects x_1 to x_2, another connects x_2' to x_3, and the third connects x_1' to x_3'. Also, $I_{3,2} \cap B_{3,2} \subseteq Q_{3,2}$, where $Q_{3,2}$ is the event that there exist $x_4, y_4 \in B(n)$ and

x_4, $y_4 \in B(n)$ and six paths such that the first path connects x_1 and x_4, the second path connects x_4 and x_2, the third path connects x_4 and x_3, the fourth path connects x'_1 and y_4, the fifth path connects y_4 and x'_2, and the last path connects y_4 and x'_3. By the v.d. Berg–Kesten inequality, then,

$$\frac{1}{|B(n)|}\,|K_n^{(3)}| \leqslant \left(\frac{2d}{p}\right)^3$$  $$+\frac{1}{2}\left(\frac{2d}{p}\right)^3$$  (3.11)

Applying the same argument, we get

$$\frac{1}{|B(n)|}\,|K_n^{(4)}| \leqslant \left(\frac{2d}{p}\right)^4 \Bigg\{ 3$$  $$+6$$ 

$$+\frac{3}{2}$$  $$+\frac{3}{2}$$  $$\Bigg\}$$ (3.12)

To estimate each Feynman diagram, we need the following result obtained by Hara and Slade[4]: There exist constants d_0 and c_0 such that the Fourier transform of $g(0, \cdot)$,

$$|\hat{g}(k)| \leqslant \frac{c_0}{|k|^2} \tag{3.13}$$

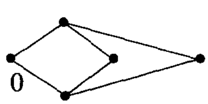
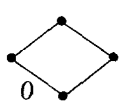
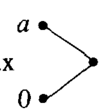
for all $k = (k_1,..., k_\alpha)$, $|k_\alpha| \leqslant \pi$, $0 < p < p_c$ and $d \geqslant d_0$.

By (3.9) and (3.13), there exists a constant c such that

$$\frac{1}{|B(n)|}\,|K''_n| \leqslant \frac{1}{2}\left(\frac{2d}{p}\right)^2 \|\hat{g}\|_2^2 < \left(\frac{1}{p}\right)^2 c$$

$0 < p < p_c$, if $d \geqslant \max(d_0, 5)$. By the monotone convergence theorem, the same estimate holds for $0 < p \leqslant p_c$ and $d \geqslant \max(d_0, 5)$.

In the third derivative (3.11), the first Feynman diagram is bounded by $\|\hat{g}\|_3^3$. The second diagram in (3.11) can be estimated by using Hölder's inequalities and translational invariance of g:

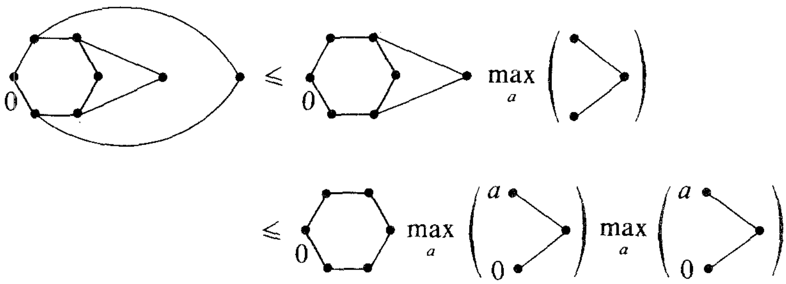
 $$\leqslant \Bigg($$  $$\Bigg)\Bigg(\max_a$$  $$\Bigg)$$ (3.14)

The right side of the above is bounded by $\|\hat{g}\|_4^4 \|\hat{g}\|_2^2$. Hence

$$\frac{1}{|B(n)|} |K_n^{(3)}| \leqslant \left(\frac{1}{p}\right)^3 c \tag{3.15}$$

for all $0 < p \leqslant p_c$ if $d \geqslant \max(d_0, 9)$.

The first diagram in (3.12) is bounded by $\|\hat{g}\|_4^4$. The second diagram in (3.12) is estimated in the same way as (3.14). It is bounded by $\|\hat{g}\|_5^5 \|\hat{g}\|_2^2$. The third diagram is estimated using Hölder's inequalities successively:



$$\leqslant \|\hat{g}\|_6^6 \|\hat{g}\|_2^2 \|\hat{g}\|_2^2 \tag{3.16}$$

The fourth diagram can be estimated in the same way. It is bounded by

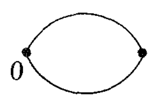
$$\|\hat{g}\|_5^5 \|\hat{g}\|_3^3 \|\hat{g}\|_2^2$$

Therefore

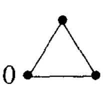
$$\frac{1}{|B(n)|} |K_n^{(4)}| \leqslant \left(\frac{1}{p}\right)^4 c \tag{3.17}$$

for all $0 < p \leqslant p_c$ if $d \geqslant \max(d_0, 13)$.

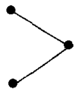
This method can be applied to estimate $|K_n^{(N)}|$ for general N. The evolution of Feynman diagrams in $K_n^{(N)}$ as N increases can be described as follows: when $N = 2$, there is a diagram

F_2: 

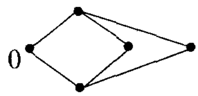
When $N = 3$, two diagams are obtained, where one is obtained by adding a vertex along an edge of F_2 to get

$F_{3,1}$: 0 

The other is obtained by adding a graph

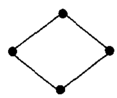
F: 

on F_2 such that two endpoints of F are attached to two different edges of F_2. This gives

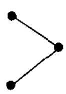
$F_{3,2}$: 

in $K_n^{(3)}$.

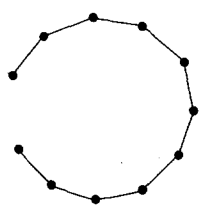
This pattern of Feynman diagrams continues to produce all possible Feynman diagrams in $K_n(N)$. In $F_{3,2}$, let us call



a level 1 graph of $F_{3,2}$ and

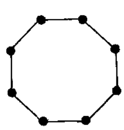


a level 2 graph of $F_{3,2}$. The order of levels is determined by the construction of $F_{3,2}$. In $K_n^{(N)}$, all Feynman diagrams are unions of at most $N-1$ levels, where each level is of the form



for level K, $K \geqslant 2$,

except the first level, which must be like

 level 1

Note that in a diagram of $K_n^{(N)}$ the number of vertices in level k, $k \geqslant 2$, is at most $2N-1$. The number of vertices in level 1 is at most $2N-2$. Now we use Hölder's inequalities successively to estimate a Feynman diagram, as illustrated in (3.16). Each time the highest level is reduced. Then the Feynman diagram is bounded by

$$\prod_{i=1}^{m} \|\hat{g}\|_{\alpha_i}^{\alpha_i} \tag{3.18}$$

where $m \leqslant N-1$, $\alpha_i \leqslant 2N-2$ for all $i = 1, 2,..., m$. It follows from (3.13) that (3.18) is bounded by a constant if $0 < p \leqslant p_c$ and $d \geqslant \max(d_0, 4N-3)$. The number of Feynman diagrams in $K_n^{(N)}$ is a combinatorial constant. Therefore, there exists a constant $c_N(d)$ such that

$$\frac{1}{|B(n)|} |K_n^{(N)}| \leqslant \left(\frac{1}{p}\right)^N c_N(d) \tag{3.19}$$

for all $0 < p \leqslant p_c$ for all n and all $d \geqslant \max(d_0, 4N-3)$. End of the proof of Theorem 1.1.

APPENDIX

For the convenience of the reader, we include here an exact solution for $\kappa_{\mathrm{B}}(p)$ in the Bethe lattice with degree $d+1$. Let $\{x_1, x_2,..., x_n\}$ be a set of sites containing the root. Observe that $\{x_1, x_2,..., x_n\}$ is the cluster containing the root if and only if there are $n-1$ open bonds connecting $x_1, x_2,..., x_n$ and $dn+1$ closed bonds attached to $x_1, x_2,..., x_n$. Let A_n be the number of clusters of size n containing the root. Let C be the cluster containing the root. Then

$$P(|C| = n) = A_n(1-p)^{dn+1} p^{n-1} \tag{A.1}$$

There,

$$\begin{aligned}\kappa_{\mathrm{B}}(p) = E\left(\frac{1}{|C|}\right) &= \sum_{n=1}^{\infty} \frac{1}{n} P(|C| = n) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} A_n(1-p)^{dn+1} p^{n-1} \\ &= \frac{1-p}{p} \sum_{n=1}^{\infty} \frac{1}{n} A_n[p(1-p)^d]^n \end{aligned}\tag{A.2}$$

For $0 < p < 1$ and $p \neq p_c = 1/(d+1)$, we may take the derivative in (A.2) to get

$$\kappa_{\mathrm{B}}'(p) = \frac{[(1-p)/p]'}{[(1-p)/p]} \kappa_{\mathrm{B}}(p) + \frac{1-(d+1)p}{(1-p)p} P(|C| < \infty) \tag{A.3}$$

The general formula for $\kappa_{\mathrm{B}}^{(n)}$ can be obtained recursively by using (A.3). It follows from (A.3) that $P(|C| < \infty) = 1$ when $p < p_c$ and induction that $\lim_{p \to p_c} \kappa_{\mathrm{B}}^{(n)}$ is bounded. Moreover, by solving the differential equation (4.3), we get

$$\kappa_B(p) = -d - (d+1)\frac{(1-p)\ln(1-p)}{p} \tag{A.4}$$

for $0 < p \leqslant 1/(d+1)$.

ACKNOWLEDGMENT

The research of W.-S. Y. was supported in part by NSF grant DMS 9096256.

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